must have a corresponding value in Eqs. (3). From the third and sixth of Eqs. (3), a solution is obtained for $\sin\theta_c$ and $\dot{\theta}_c$:

$$\sin\theta_c = \frac{\omega A_Z(J_3/J_2) - B_Z}{J_1(J_3/J_2) - J_2} \tag{6}$$

$$\dot{\theta}_c = \frac{B_Z - J_2\sin\theta_c}{J_3\cos\theta_c} \tag{7}$$

Similarly, a solution is obtained for $\sin\psi_c$ and $\dot{\psi}_c$ from the second and fifth of Eqs. (3):

$$\sin\psi_c = \frac{1}{\cos\theta_c}\,\frac{\omega A_y(J_3/J_2) - B_y}{J_1(J_3/J_2) - J_2} \tag{8}$$

$$\dot{\psi}_c = \frac{\omega A_y + (\dot{\theta}_c J_2\sin\theta_c - J_1\cos\theta_c)\sin\psi_c}{J_2\cos\theta_c\cos\psi_c} \tag{9}$$

The next logical step in this development is to solve for τ and ξ, using the first and fourth of Eqs. (3) in combination with Eqs. (6–9). Unfortunately, this step cannot be carried through in a straightforward manner. The resulting equations for ξ and τ require the simultaneous solution of two transcendental equations.

Under laboratory conditions, this solution is practical with the aid of a high-speed computer. However, the system being developed here is intended for operational use under anything but laboratory conditions. If the solution did not, for some reason, converge properly under an off-design condition, there would not be time to force convergence nor would there be time to work out new initial values that might result in convergence.

Rather than attempt the solution just described, it is suggested that two approximations for ξ and τ be used. These will result in trajectories that are not flown under constant thrust. However, the approximations are sufficiently accurate to assure that the throttling range required to accommodate the deficiencies in the solution will be small, on the order of 2%. This has been verified with a two-degree of-freedom digital simulation, using Eqs. (6, 7, 10, and 11) for trajectory control.

The approximations for ξ and τ are given as

$$\xi = 1 - \exp\left[-\frac{1}{V_j}\left(|\dot{\mathbf{S}}| + 2g\,\frac{h - h_t}{V + V_t}\right)\right] \qquad (V + V_t) > 0 \tag{10}$$

$$\tau = \left|\frac{S^2}{\dot{\mathbf{S}}\cdot\mathbf{S}\left[\dfrac{1}{\ln(1-\xi)} + \dfrac{1}{\xi}\right] + \dot{\mathbf{r}}_t\cdot\mathbf{S}}\right|\frac{1}{K_n} \tag{11}$$

where $\mathbf{S} = \mathbf{r} - \mathbf{r}_t$. The K term in Eq. (11) is found in the following manner: assume that the first time a τ is computed during a flight, $K_{(n-1)} = 1.0$. Compute the thrust level required to satisfy Eqs. (10) and (11) from Eq. (12):

$$F_c = V_j m(\xi/\tau) \tag{12}$$

Now compute $\theta_c, \dot{\theta}_c, \psi_c$, and $\dot{\psi}_c$ from the appropriate equations, and then compute the predicted landing point from Eqs. (1) and (3). A predicted slant range vector is computed as

$$\mathbf{S}_t = \mathbf{r} - \mathbf{r}_t \tag{13}$$

Then K_n is found from Eq. (14):

$$K_n = \left(\frac{\mathbf{S}_t\cdot\mathbf{S}_t}{\mathbf{S}\cdot\mathbf{S}}\right)^{1/2} \tag{14}$$

The K term need not be computed continuously. For most flights, updating every 10 sec is sufficient. The results from the digital simulation and the associated error analysis of this concept are to be reported at a later date.

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[1] Markson, E., Bryant, J., and Bergsten, F., "Simulation of manned lunar landing," *AIAA Progress in Astronautics and Rocketry: Technology of Lunar Exploration*, edited by Clifford I. Cummings and Harold R. Lawrence (Academic Press Inc., New York, 1963), pp. 561–589.

Contour Calculations for Chemical Nonequilibrium Flow

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The problem of a reacting flow expanding through a divergent nozzle has been examined in the light of designing nozzle contours to produce specific interrelationships among the problem variables. Using a Pace Electronic analog computer, solutions have been generated for a variety of parameter restrictions. The results show that hydrodynamically reasonable nozzle shapes may be used to provide a convenient method of experimentally producing arbitrary supersonic nonequilibrium flows.

Nomenclature

a = reciprocal of dimensionless area A
E = dissociation energy
F = dimensionless recombination rate, $4r_0p_0^2\eta k_R/u_0R^2T_0^2T$
G = dimensionless equilibrium constant, $K_p/4\eta p_0T$
k_R = recombination rate
K = dimensionless term, $4NE_A/RT_0$
K_p = equilibrium constant
M = molecular weight
N = Avogadro's number
p = dimensionless pressure
r = dimensionless streamline coordinate
R = universal gas constant
T = dimensionless temperature
u = dimensionless velocity
α = mass fraction of dissociated atomic species
β = critical throat parameter, $\rho_0u_0^2/p_0$
Δ = computer time scale factor
η = dimensionless constant, $\beta RT_0/2M_Au_0^2$
ρ = dimensionless mass density
τ = computer time

Subscripts

A = atomic
e = local equilibrium
0 = dimensional throat value

IN this note a method for designing hypersonic nozzle contours for specified thermodynamic requirements is presented using the governing equations of pseudo-one-dimensional inviscid flow of an ideal diatomic gas undergoing a dissociation reaction. For nonequilibrium flow, where chemical reactions proceed at finite rates, the flow properties are dependent upon the axial distance traveled. As a result, it is

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possible to determine unique nozzle area profiles for particular nonequilibrium flow processes. Since the reaction equation is nonlinear and the set of equations is closely coupled, there is no simple closed-form analytical solution.

Although the general formulation of the problem may be extended to a mixture of gases, e.g., Ref. 1. only a diatomic model was used to demonstrate the method presented. The assumptions made with regard to the gas model are as follows:

1) The fluid is a thermally perfect pure diatomic gas. Viscosity and diffusion are considered to be negligible.

2) The flow is adiabatic.

3) The contribution from energy in electronic and ionization states is insignificant for the range of temperatures and densities considered.

4) The vibrational energy of the molecular species is approximately linear with respect to temperature.

5) Chemical relaxation is the only nonequilibrium effect that occurs.

All numerical calculations were based upon oxygen as the fluid media. Nozzle contours were determined for initial conditions of 5000°K, 20 atm, mass fraction of dissociated species 0.800, and an equilibrium mass fraction of dissociated species 0.646.

The one-dimensional flow of a dissociating ideal gas can be described by the equations of mass continuity (both global and species), momentum and energy conservation, and the equation of state. The rate of composition change can be related to the axial distance traveled so that the simultaneous solution of this set of governing equations will yield the gasdynamic variables, including the flow area, as a function of axial length. Since the number of equations is one less than the number of variables, one additional condition, normally the nozzle contour but in this case a thermodynamic condition, must be specified to complete the definition of the problem.

In addition to the basic equations just discussed, there are a number of auxiliary equations interrelating the thermodynamic and kinetic characteristics of the problem. The statistical mechanical basis for equilibrium properties is given by Cambel et al.,[2] and, in general, the Lighthill ideal dissociating gas approximations[3] are used for this problem. In this analysis a linear temperature dependence of the vibrational energy was used rather than a constant value as suggested by Lighthill. Also, the form of the functional dependence of the equilibrium constant on temperature differs somewhat from Lighthill's approximation and is more accurate. This accuracy is a result of the fact that with the method of solution employed the value of the equilibrium constant is set at the initial conditions, and the only errors are those small deviations that arise in the process of generating a solution.

Recombination rates in oxygen have been determined experimentally by measuring dissociation rates and then inferring recombination rates from calculated values of the equilibrium constant. However, this approach is highly sensitive to temperature, and no real agreement exists at this time. For purposes of this study, the recombination rate was assumed to vary inversely with the temperature, as suggested by Rink et al.[4]

Basically, the analog computer solves initial condition problems wherein all the boundary conditions can be set at time zero, and then the solution is generated as a function of time. In this application to a nozzle flow, considerable care must be taken to formulate the problem properly, since a nozzle flow is not an initial condition problem but truly a boundary-value problem since the exit pressure determines whether the expansion is subsonic or supersonic. For the solutions presented here, this difficulty was avoided by starting at the throat and forcing the solution along the supersonic leg. Ideally, the equations should be cast in a self-compensating form, that is, for a decreasing function the derivative should be equal to the negative of the variable times a positive function. This formulation has a tendency to correct itself in that, if the variable is too large, it adjusts the derivative to compensate. Effectively, this approach forces all errors to oscillate around the proper solution rather than continually increase as the solution progresses. Often this form may be affected by replacing a natural variable by an artificial one such as the reciprocal or logarithm or else a combination of natural variables.

Since the temperature occurs most often in the set of equations to be solved, it was chosen as the independent variable and related to the machine time by the relation

$$(d/d\tau)\ (1/T) = 1/\Delta \tag{1}$$

After eliminating the density by using the global continuity equation, the remaining equations become

$$\frac{dp}{d\tau} = -\beta a \frac{du}{d\tau} \tag{2}$$

$$\frac{1}{a}\frac{da}{d\tau} = \frac{1}{p}\frac{dp}{d\tau} - \frac{T}{\Delta} + \frac{1}{1+\alpha}\frac{d\alpha}{d\tau} - \frac{1}{u}\frac{du}{d\tau} \tag{3}$$

$$u\frac{du}{d\tau} = \frac{\eta}{2\beta}\left[(9+\alpha)\frac{T^2}{\Delta} - (K+T)\frac{d\alpha}{d\tau}\right] \tag{4}$$

$$\frac{d\alpha}{d\tau} = \frac{Fp}{u}\left[G(1-\alpha) - \frac{a\alpha^2}{u}\right]\frac{dr}{d\tau} \tag{5}$$

$$\frac{dF}{d\tau} = 2F\frac{T}{\Delta} \tag{6}$$

$$\frac{dG}{d\tau} = -\frac{KG}{2\Delta} \tag{7}$$

All of the variables have been nondimensionalized with respect to their value at the nozzle throat, and all of the equations have been written in differential form to take advantage of the characteristics of the analog computer.

The initial dissociated mass fraction was chosen greater than the equilibrium value as a matter of convenience in starting the computer solution. The nozzle throat is designated as the initial condition, and solutions are generated from the throat through the supersonic expansion section. Under the assumption that the nozzle contour is continuous, the throat may be defined as the point of minimum area; β is referred to as the critical throat parameter because its magnitude determines the critical throat velocity:

$$u_0 = [\beta(p_0/\rho_0)]^{1/2} \tag{8}$$

For the classical gasdynamic nozzle, the throat velocity is identical to the sonic velocity, and for this special case β must equal the ratio of specific heats. In a relaxing medium, however, it is questionable that a conventional sonic velocity is reached exactly at the throat, and thus β must represent this characteristic. In this problem β will be considered only as a constant defining the throat condition. Both the initial value of the recombination rate and the length scale factor need not be specified independently, since only their product is required. This product is chosen arbitrarily so that $F_0 = 1$.

The contours A vs r and the gasdynamical properties as a function of axial distance for two different restrictions are shown in Figs. 1 and 2. Since the nozzle throat area does not appear in the governing equations, the magnitude of the throat area is not explicitly determined. Although the axial scale is fixed by the value of F_0, the area scale is dependent upon the mass flow rate. As a consequence, the nondimensional contours are not necessarily geometrically similar to actual nozzle contours. Nevertheless, they are indicative of the relative change in flow direction along the nozzle wall. The selected mass flow for the nozzle must permit semidivergence angles within acceptable hydrodynamic limits if the initial conditions are to be physically reasonable.

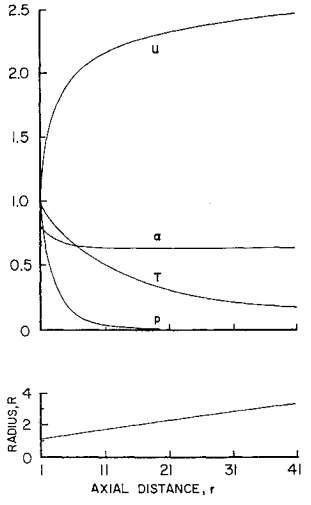

Fig. 1 Nozzle profiles; $(r - 1) = 10\,(1/T - 1)$.

The initial studies of nonequilibrium flow in nozzles was done by Bray,[5] who found that, as the flow temperature dropped, the composition quickly froze as a result of the large decrease in reaction rate. In this problem the same effect is caused by the rapid drop of the Fp/u term as the solution progresses. Since all coupling between the chemical effects and the thermodynamic variables occurs through the reaction equation, the solutions in general do not proceed past the point at which the flow composition freezes. This problem, however, does not exist if the nozzle distance is specified as a function of temperature, as shown on Fig. 1. In this case the nozzle distance is an inverse function of the temperature, and the fact that the rate of change of r was initially large accounts for a relatively gradual change in area near the nozzle throat. An almost conical nozzle with a very short transition at the throat results.

The mass fraction decaying as an inverse temperature function is shown in Fig. 2. Additional results for a variety of cases are given by Widmer.[6] Although these classes of solutions are restricted to that portion of the nozzle prior to the freezing point, the coupling terminates at this point, and there can be no further advantages obtained from geometrical considerations throughout the remainder of the expansion.

The results shown in Figs. 1 and 2 show that the electronic analog computer is a versatile nozzle simulator that conveniently facilitates the variation of specified processes and the rapid exploration through a wide spectrum of flow parameters. Comparison between specified design properties and those actually measured in a prototype nozzle system might afford new insight into the mechanism of real gas flow. In addition, the precise simulation of extreme flight conditions may be obtained more readily in the laboratory by proper nozzle design, although, undoubtedly, viscous effects would have to be considered also.

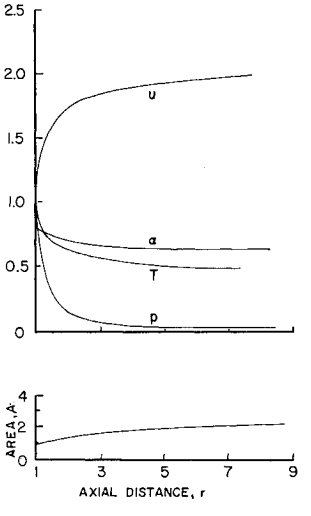

Fig. 2 Nozzle profiles; $(\alpha_0 - \alpha) = 0.15\,(1/T - 1)$.

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Axisymmetrical Turbulent Jet: Tollmien's Problem Extended

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MANY years ago, Tollmien[1, 2] solved the boundary-layer equations for the case of an axisymmetrical jet of incompressible, Newtonian fluid in isothermal, turbulent flow, using Prandtl's mixing-length theory. The case is of considerable interest in the experimental study of turbulence. Recently, a generalized version of Tollmien's equation was solved[3] in a different context, using numerical techniques different from those of Tollmien. Occasion was taken to recompute Tollmien's case, extending considerably the number of points for which numerical results are given, as well as obtaining results for the axial velocity when the kinematic momentum is known and for the axial velocity gradients for which no data had been available. Also, some of the approximations to the accurate boundary-layer solutions made in the earlier work were avoided, and it was thus believed that the present solution would be somewhat more accurate.

The boundary-layer equations for axisymmetrical, incompressible, isothermal flow of a Newtonian fluid are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{1}{\rho}\frac{\partial \tau_{x,r}}{\partial r} \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial (vr)}{\partial r} = 0 \tag{2}$$

where x is the axial coordinate, r is the radial coordinate, and u and v are the velocity components in the two directions, respectively. The boundary conditions are

$$r = 0 \qquad v = 0 \tag{3a}$$

$$r = 0 \qquad \partial u/\partial r = 0 \tag{3b}$$

$$r = \infty \qquad u = 0 \tag{3c}\dagger$$

Equations (1) and (2) are transformed as follows: assuming Prandtl's mixing-length hypothesis,

$$\frac{\tau_{x,r}}{\rho} = B\left|\frac{\partial u}{\partial r}\right|\frac{\partial u}{\partial r} \tag{4}$$

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† Tollmien assumed a *finite* boundary-layer thickness η_R, which, by implication, led to a discontinuity in F''' at $\eta = \eta_R$.